

HIGHER ORDER PARALLEL SURFACES IN BIANCHI-CARTAN-VRANCEANU SPACES

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ABSTRACT. We give a full classification of higher order parallel surfaces in three-dimensional homogeneous spaces with four-dimensional isometry group, i.e. in the so-called Bianchi-Cartan-Vranceanu family. This gives a positive answer to a conjecture formulated in [2]. As a partial result, we prove that totally umbilical surfaces only exist if the ambient Bianchi-Cartan-Vranceanu space is a Riemannian product of a surface of constant Gaussian curvature and the real line, and we give a local parametrization of all totally umbilical surfaces.

1. INTRODUCTION

A Riemannian manifold (M, g) is said to be homogeneous if for every two points p and q in M , there exists an isometry of M , mapping p into q . The classification of simply connected 3-dimensional homogeneous spaces is well-known. The dimension of the isometry group must equal 6, 4 or 3. If the isometry group is of dimension 6, M is a complete real space form, i.e. Euclidean space \mathbb{E}^3 , a sphere $\mathbb{S}^3(\kappa)$, or a hyperbolic space $\mathbb{H}^3(\kappa)$. If the dimension of the isometry group is 4, M is isometric to $SU(2)$, the special unitary group, to $[SL(2, \mathbb{R})]^\sim$, the universal covering of the real special linear group, to Nil_3 , the Heisenberg group, all with a certain left-invariant metric, or to a Riemannian product $S^2(\kappa) \times \mathbb{R}$ or $\mathbb{H}^2(\kappa) \times \mathbb{R}$. Finally, if the dimension of the isometry group is 3, M is isometric to a general simply connected Lie group with left-invariant metric. As will become clear in the next section, Bianchi-Cartan-Vranceanu spaces are in fact the spaces with 4-dimensional isometry group mentioned above, together with \mathbb{E}^3 and $\mathbb{S}^3(\kappa)$.

The classification above contains the eight “model geometries” appearing in the famous conjecture of Thurston on the classification of 3-manifolds, namely \mathbb{E}^3 , \mathbb{S}^3 , \mathbb{H}^3 , $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $[SL(2, \mathbb{R})]^\sim$, Nil_3 and Sol_3 . See for example [18]. In theoretical cosmology, the metrics on these spaces are known as Bianchi-Kantowski-Sachs type metrics, used to construct spatially homogeneous spacetimes, see for example [12].

Immersion of curves and surfaces in 3-dimensional real space forms are extensively studied and it is now very natural to allow the other 3-dimensional homogeneous manifolds as ambient spaces. Initial work in this direction can be found in [11] and [6].

An important class of surfaces to study are parallel surfaces. These immersions have a parallel second fundamental form and hence their extrinsic invariants “are the same” at every point. Parallel submanifolds in real space forms are classified in [1]. In [7], [8], [9] and [14], the notion of higher order parallelism is introduced and a classification for hypersurfaces in real space forms is obtained. In [2], a classification of parallel surfaces in 3-dimensional homogeneous spaces with 4-dimensional isometry group is given, whereas the classification of higher order parallel surfaces is formulated as a conjecture. In this article we will prove this conjecture (Theorem 8). For an overview of the theory of parallel and higher order parallel submanifolds we refer to [15].

Another important class of surfaces are totally umbilical ones. From an extrinsic viewpoint, these surfaces are curved equally in every direction. We will give a full local classification of totally umbilical surfaces in 3-dimensional homogeneous spaces with 4-dimensional isometry group

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(Theorems 5, 6 and 7). Although in a real space form a totally umbilical surface is automatically parallel, this will no longer be the case in the spaces under consideration.

2. EXAMPLES OF THREE-DIMENSIONAL HOMOGENEOUS SPACES

2.1. The Heisenberg group Nil_3 with left-invariant metric. The Heisenberg group Nil_3 is a Lie group which is diffeomorphic to \mathbb{R}^3 and the group operation is defined by

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = \left(x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{x\bar{y}}{2} - \frac{\bar{x}y}{2} \right).$$

Remark that the mapping

$$\text{Nil}_3 \rightarrow \left\{ \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right) : (x, y, z) \mapsto \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

is an isomorphism between Nil_3 and a subgroup of $\text{GL}(3, \mathbb{R})$. For every non-zero real number τ the following metric on Nil_3 is left-invariant:

$$ds^2 = dx^2 + dy^2 + 4\tau^2 \left(dz + \frac{y dx - x dy}{2} \right)^2.$$

After the change of coordinates $(x, y, 2\tau z) \mapsto (x, y, z)$, this metric is expressed as

$$(1) \quad ds^2 = dx^2 + dy^2 + (dz + \tau(y dx - x dy))^2.$$

2.2. The projective special linear group $\text{PSL}(2, \mathbb{R})$ with left-invariant metric. Consider the following subgroup of $\text{GL}(2, \mathbb{R})$:

$$\text{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}.$$

First note that this group is isomorphic to the following subgroup of $\text{GL}(2, \mathbb{C})$:

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\},$$

via the isomorphism

$$\text{SL}(2, \mathbb{R}) \rightarrow G : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}.$$

Now consider the Poincaré disc-model for the hyperbolic plane $\mathbb{H}^2(\kappa)$ of constant Gaussian curvature $\kappa < 0$:

$$(2) \quad \begin{aligned} \mathbb{H}^2(\kappa) &\cong \left(\left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < -\frac{4}{\kappa} \right\}, \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} \right) \\ &\cong \left(\left\{ z \in \mathbb{C} \mid |z|^2 < -\frac{4}{\kappa} \right\}, \frac{dz d\bar{z}}{(1 + \frac{\kappa}{4}|z|^2)^2} \right) \end{aligned}$$

and define

$$F_{\left(\frac{\alpha}{\beta} \frac{\beta}{\bar{\alpha}}\right)}(z) = \frac{2}{\sqrt{-\kappa}} \frac{\alpha \sqrt{-\kappa} z + 2\beta}{\beta \sqrt{-\kappa} z + 2\bar{\alpha}}.$$

Note that for $\kappa = -4$, this Möbius transformation simplifies to $z \mapsto \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}$. The mapping

$$G \times \mathbb{H}^2(\kappa) \rightarrow \mathbb{H}^2(\kappa) : (A, z) \mapsto F_A(z)$$

is a transitive, isometric action with stabilizers isomorphic to the circle group $\text{SU}(1) = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$. This action induces the following transitive action on the unitary tangent bundle $\text{U}\mathbb{H}^2(\kappa)$:

$$(3) \quad G \times \text{U}\mathbb{H}^2(\kappa) \rightarrow \text{U}\mathbb{H}^2(\kappa) : (A, (z, v)) \mapsto (F_A(z), (F_A)_* v),$$

with stabilizers of order two. Hence we can identify $\mathbb{UH}^2(\kappa)$ with

$$\mathrm{PSL}(2, \mathbb{R}) = \frac{\mathrm{SL}(2, \mathbb{R})}{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}}.$$

Let us now define a metric on $\mathbb{UH}^2(\kappa)$. If $\gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{UH}^2(\kappa) : t \mapsto (z(t), v(t))$ is a curve, with $z(t)$ a curve in $\mathbb{H}^2(\kappa)$ and for every $t \in I$, $v(t) \in T_{z(t)}\mathbb{H}^2(\kappa)$ and $\|v(t)\| = 1$, we put

$$(4) \quad \|\gamma'(t_0)\|^2 = \|z'(t_0)\|^2 + \left(\frac{2\tau}{\kappa}\right)^2 \|(\nabla_{z'}v)_{z(t_0)}\|^2, \quad \tau \in \mathbb{R} \setminus \{0\},$$

where $\nabla_{z'}v$ is the covariant derivative of the vector field v along the curve $z(t)$. For $\tau = \pm \frac{\kappa}{2}$, this metric is induced from the standard metric on the tangent bundle. By varying the parameter τ , we distort the length of the fibres. It is clear that the action (3) is now isometric and hence the induced metric on $\mathrm{PSL}(2, \mathbb{R})$ via the identification is left-invariant. The metric (4) can be explicitly computed, analogous as in [6], in the coordinate system

$$\begin{aligned} \mathbb{D}^2\left(\frac{2}{\sqrt{-\kappa}}\right) \times \mathbb{S}^1(1) &\rightarrow \mathbb{UH}^2(\kappa) : \\ ((x, y), \theta) &\mapsto \left((x, y), \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right) \left(\cos\left(\frac{\kappa}{2\tau}\theta\right) \frac{\partial}{\partial x} + \sin\left(\frac{\kappa}{2\tau}\theta\right) \frac{\partial}{\partial y} \right) \right), \end{aligned}$$

where $\mathbb{D}^2\left(\frac{2}{\sqrt{-\kappa}}\right)$ is the disc of radius $\frac{2}{\sqrt{-\kappa}}$, yielding the following result:

$$(5) \quad ds^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^2} + \left(d\theta + \tau \frac{y dx - x dy}{1 + \frac{\kappa}{4}(x^2 + y^2)}\right)^2, \quad \kappa < 0.$$

2.3. The special orthogonal group $\mathrm{SO}(3)$ with left-invariant metric. Consider the following subgroup of $\mathrm{GL}(2, \mathbb{C})$:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Using stereographic projection, we have for an arbitrary $\kappa > 0$:

$$(6) \quad \mathbb{S}^2(\kappa) \setminus \{\infty\} \cong \left(\mathbb{R}^2, \frac{dx^2 + dy^2}{\left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^2} \right) \cong \left(\mathbb{C}, \frac{dz d\bar{z}}{\left(1 + \frac{\kappa}{4}|z|^2\right)^2} \right).$$

The analogy with the previous case is clear and we could now proceed in the same way as above, putting

$$F_{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}}(z) = \frac{2}{\sqrt{\kappa}} \frac{\alpha\sqrt{\kappa}z + 2\beta}{(-\bar{\beta}\sqrt{\kappa}z + 2\bar{\alpha})},$$

and being careful in calculations involving the symbol ∞ . In this way we would find that $\mathrm{US}^2(\kappa)$ can be identified with

$$\mathrm{PSU}(2) = \frac{\mathrm{SU}(2)}{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}}.$$

But since $\mathrm{PSU}(2)$ is isomorphic to $\mathrm{SO}(3)$, see for example [19], there is an easier way to construct the desired group action. Looking at $\mathbb{S}^2(\kappa)$ as a hypersphere in \mathbb{E}^3 centered at the origin, we can identify both points of the surface and tangent vectors to it with elements of \mathbb{R}^3 and we define

$$\mathrm{SO}(3) \times \mathrm{US}^2(\kappa) \rightarrow \mathrm{US}^2(\kappa) : (A, (p, v)) \mapsto (Ap, Av).$$

This is a transitive action with trivial stabilizers and a metric on $\mathrm{US}^2(\kappa)$ analogous to (4) turns it into an isometric action. This means that the induced metric on $\mathrm{SO}(3)$ will be left-invariant and in the local coordinates

$$\mathbb{R}^2 \times \mathbb{S}^1(1) \rightarrow \mathrm{US}^2(\kappa) : ((x, y), \theta) \mapsto \left((x, y), \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right) \left(\cos\left(\frac{\kappa}{2\tau}\theta\right) \frac{\partial}{\partial x} + \sin\left(\frac{\kappa}{2\tau}\theta\right) \frac{\partial}{\partial y} \right) \right)$$

it is expressed as

$$(7) \quad ds^2 = \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \left(d\theta + \tau \frac{y dx - x dy}{1 + \frac{\kappa}{4}(x^2 + y^2)} \right)^2, \quad \kappa > 0.$$

2.4. The Riemannian product spaces $\mathbb{H}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{R}$. Using respectively the models (2) and (6) for $\mathbb{H}^2(\kappa)$ and $\mathbb{S}^2(\kappa)$, one sees that the Riemannian product metric on these spaces can be expressed (locally) as

$$(8) \quad ds^2 = \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + dz^2.$$

2.5. Bianchi-Cartan-Vranceanu spaces. Remark that the metrics (1), (5), (7) and (8) of the homogeneous spaces above are of the same type. Cartan classified all 3-dimensional spaces with 4-dimensional isometry group in [5]. In particular, he proved that they are all homogeneous and obtained the following two-parameter family of spaces, which are now known as the *Bianchi-Cartan-Vranceanu spaces* or *BCV spaces* for short. For $\kappa, \tau \in \mathbb{R}$, we define $\widetilde{M}^3(\kappa, \tau)$ as the following open subset of \mathbb{R}^3 :

$$\left\{ (x, y, z) \in \mathbb{R}^3 \mid 1 + \frac{\kappa}{4}(x^2 + y^2) > 0 \right\},$$

equipped with the metric

$$(9) \quad ds^2 = \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} + \left(dz + \tau \frac{y dx - x dy}{1 + \frac{\kappa}{4}(x^2 + y^2)} \right)^2.$$

See also [3], [4] and [20]. The result of Cartan shows that the examples above cover in fact all possible 3-dimensional homogeneous spaces with 4-dimensional isometry group. The BCV family also includes two real space forms, which have 6-dimensional isometry group. The full classification of these spaces is as follows:

- if $\kappa = \tau = 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \mathbb{E}^3$;
- if $\kappa = 4\tau^2 \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \mathbb{S}^3(\frac{\kappa}{4}) \setminus \{\infty\}$;
- if $\kappa > 0$ and $\tau = 0$, then $\widetilde{M}^3(\kappa, \tau) \cong (\mathbb{S}^2(\kappa) \setminus \{\infty\}) \times \mathbb{R}$;
- if $\kappa < 0$ and $\tau = 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \mathbb{H}^2(\kappa) \times \mathbb{R}$;
- if $\kappa > 0$ and $\tau \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong [U(\mathbb{S}^2(\kappa) \setminus \{\infty\})]^\sim \cong \text{SU}(2) \setminus \{\infty\}$;
- if $\kappa < 0$ and $\tau \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong [U\mathbb{H}^2(\kappa)]^\sim \cong [\text{SL}(2, \mathbb{R})]^\sim$;
- if $\kappa = 0$ and $\tau \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \text{Nil}_3$.

To end this section, we discuss the geometry of these spaces. The following vector fields form an orthonormal frame on $\widetilde{M}^3(\kappa, \tau)$:

$$e_1 = \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right) \frac{\partial}{\partial x} - \tau \frac{\partial}{\partial z}, \quad e_2 = \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right) \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.$$

It is clear that these vector fields satisfy the following commutation relations:

$$(10) \quad [e_1, e_2] = -\frac{\kappa}{2}ye_1 + \frac{\kappa}{2}xe_2 + 2\tau e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

The Levi Civita connection of $\widetilde{M}^3(\kappa, \tau)$ can then be computed using Koszul's formula:

$$(11) \quad \begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= \frac{\kappa}{2}ye_2, & \widetilde{\nabla}_{e_1} e_2 &= -\frac{\kappa}{2}ye_1 + \tau e_3, & \widetilde{\nabla}_{e_1} e_3 &= -\tau e_2, \\ \widetilde{\nabla}_{e_2} e_1 &= -\frac{\kappa}{2}xe_2 - \tau e_3, & \widetilde{\nabla}_{e_2} e_2 &= \frac{\kappa}{2}xe_1, & \widetilde{\nabla}_{e_2} e_3 &= \tau e_1, \\ \widetilde{\nabla}_{e_3} e_1 &= -\tau e_2, & \widetilde{\nabla}_{e_3} e_2 &= \tau e_1, & \widetilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

Remark that $\tilde{\nabla}_X e_3 = \tau(X \times e_3)$ for every $X \in T\tilde{M}^3(\kappa, \tau)$, where the cross product is defined as an anti-symmetric bilinear operation, satisfying $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$. The equations in (11) yield the following expression for the curvature tensor of $\tilde{M}^3(\kappa, \tau)$:

$$(12) \quad \tilde{R}(X, Y)Z = (\kappa - 3\tau^2)(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ - (\kappa - 4\tau^2)(\langle Y, e_3 \rangle \langle Z, e_3 \rangle X - \langle X, e_3 \rangle \langle Z, e_3 \rangle Y + \langle X, e_3 \rangle \langle Y, Z \rangle e_3 - \langle Y, e_3 \rangle \langle X, Z \rangle e_3)$$

for $p \in \tilde{M}^3(\kappa, \tau)$ and $X, Y, Z \in T_p \tilde{M}^3(\kappa, \tau)$.

Consider the following Riemannian surface with constant Gaussian curvature κ :

$$\tilde{M}^2(\kappa) = \left(\left\{ (x, y) \in \mathbb{R}^2 \mid 1 + \frac{\kappa}{4}(x^2 + y^2) > 0 \right\}, \frac{dx^2 + dy^2}{(1 + \frac{\kappa}{4}(x^2 + y^2))^2} \right).$$

Then the mapping

$$\pi : \tilde{M}^3(\kappa, \tau) \rightarrow \tilde{M}^2(\kappa) : (x, y, z) \mapsto (x, y)$$

is a Riemannian submersion, referred to as the *Hopf-fibration*. For $\kappa = 4\tau^2 \neq 0$, this mapping coincides with the “classical” Hopf-fibration $\pi : \mathbb{S}^3(\frac{\kappa}{4}) \rightarrow \mathbb{S}^2(\kappa)$. In the following, by a *Hopf-cylinder* we mean the inverse image of a curve in $\tilde{M}^2(\kappa)$ under π . By a *leaf* of the Hopf-fibration, we mean a surface which is everywhere orthogonal to the fibres. From Frobenius’ theorem and (10), it is clear that this only exists if $\tau = 0$.

3. SURFACES IMMERSED IN BCV SPACES

Let us start with recalling the basic formulas from the theory of submanifolds. Suppose that $F : M^n \rightarrow \tilde{M}^{n+k}$ is an isometric immersion of Riemannian manifolds and denote by ∇ the Levi Civita connection of M^n and by $\tilde{\nabla}$ that of \tilde{M}^{n+k} . With the appropriate identifications, the formulas of Gauss and Weingarten state respectively

$$(13) \quad \tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y),$$

$$(14) \quad \tilde{\nabla}_X \xi = -S_\xi X + \nabla_X^\perp \xi,$$

where X and Y are vector fields tangent to M^n and ξ is a normal vector field along M^n . The symmetric (1,2)-tensor field α , taking values in the normal bundle, is called the *second fundamental form*, the symmetric (1,1)-tensor field S_ξ on M^n is the *shape operator associated to ξ* and ∇^\perp is a connection in the normal bundle. From these formulas the equations of Gauss and Codazzi can be deduced:

$$(15) \quad \tan(\tilde{R}(X, Y)Z) = R(X, Y)Z + S_{\alpha(X, Z)}Y - S_{\alpha(Y, Z)}X,$$

$$(16) \quad \tan(\tilde{R}(X, Y)\xi) = (\nabla_Y S)_\xi X - (\nabla_X S)_\xi Y,$$

for $p \in M^n$ and $X, Y, Z \in T_p M^n$, $\xi \in T_p^\perp M^n$. Here R is the Riemann-Christoffel curvature tensor of M^n , \tilde{R} that of \tilde{M}^{n+k} , “tan” denotes the projection on the tangent space to M^n and $(\nabla_X S)_\xi Y = \nabla_X(S_\xi Y) - S_\xi(\nabla_X Y) - S_{\nabla_X^\perp \xi} Y$.

Now let $F : M^2 \rightarrow \tilde{M}^3(\kappa, \tau)$ be an isometric immersion of an oriented surface in a BCV space, with unit normal ξ and associated shape operator S . We denote by θ the angle between e_3 and ξ and by T the projection of e_3 on the tangent plane to M^2 , i.e. the vector field T on M^2 such that $F_* T + \cos \theta \xi = e_3$. If we work locally, we may assume $\theta \in [0, \frac{\pi}{2}]$. The equations of Gauss (15) and Codazzi (16) give respectively

$$(17) \quad R(X, Y)Z = (\kappa - 3\tau^2)(\langle Y, Z \rangle X - \langle X, Z \rangle Y) - (\kappa - 4\tau^2)(\langle Y, T \rangle \langle Z, T \rangle X - \langle X, T \rangle \langle Z, T \rangle Y \\ + \langle X, T \rangle \langle Y, Z \rangle T - \langle Y, T \rangle \langle X, Z \rangle T) + \langle SY, Z \rangle SX - \langle SX, Z \rangle SY$$

and

$$(18) \quad \nabla_X SY - \nabla_Y SX - S[X, Y] = (\kappa - 4\tau^2) \cos \theta (\langle Y, T \rangle X - \langle X, T \rangle Y)$$

for $p \in M^2$ and $X, Y, Z \in T_p M^2$. From (17) it follows moreover that the Gaussian curvature of M^2 is given by

$$(19) \quad K = \det S + \tau^2 + (\kappa - 4\tau^2) \cos^2 \theta.$$

Finally, we remark that the following structure equations hold for $p \in M^2$ and $X \in T_p M^2$:

$$(20) \quad \nabla_X T = \cos \theta (SX - \tau JX),$$

$$(21) \quad X[\cos \theta] = -\langle SX - \tau JX, T \rangle,$$

where J denotes the rotation over $\frac{\pi}{2}$ in $T_p M^2$. These equations can be verified straightforwardly by comparing the tangential and normal components of both sides of the equality $\tilde{\nabla}_X(T + \cos \theta \xi) = \tau(X \times (T + \cos \theta \xi))$.

The following theorem is proven in [6]:

Theorem 1. [6] *Let M^2 be a simply connected, oriented Riemannian surface with metric $\langle \cdot, \cdot \rangle$, Levi Civita connection ∇ and curvature tensor R . Let J denote the rotation over $\frac{\pi}{2}$ in TM^2 and S a field of symmetric operators on TM^2 . Finally, let T be a vector field on M^2 and let $\cos \theta$ be a differentiable function, satisfying $\langle T, T \rangle + \cos^2 \theta = 1$. Then there exists an isometric immersion F of M^2 in $\tilde{M}^3(\kappa, \tau)$ with unit normal ξ , such that S is the shape operator and $e_3 = F_*T + \cos \theta \xi$ if and only if the equations (17), (18), (20) and (21) are satisfied. In this case the immersion is moreover unique up to a global isometry of $\tilde{M}^3(\kappa, \tau)$, preserving both the orientations of the base space $\tilde{M}^2(\kappa)$ and the fibres of π .*

4. PARALLEL, SEMI-PARALLEL AND HIGHER ORDER PARALLEL HYPERSURFACES

Let $F : M^n \rightarrow \tilde{M}^{n+1}$ be an isometric immersion of Riemannian manifolds and $p \in M^n$. If α is the second fundamental form and ξ is a unit normal vector field on the hypersurface, we define the scalar valued second fundamental form h to be the $(0,2)$ -tensor field satisfying $\alpha(X, Y) = h(X, Y) \xi$ for all $p \in M^n$ and $X, Y \in T_p M^n$. The covariant derivative of h is defined by

$$(\nabla h)(X, Y, Z) = X[h(Y, Z)] - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for all $X, Y, Z \in T_p M^n$ with ∇ the Levi Civita connection of M^n . If R is the curvature tensor of M^n , we also define

$$(R \cdot h)(X, Y, Z_1, Z_2) = -h(R(X, Y)Z_1, Z_2) - h(Z_1, R(X, Y)Z_2),$$

for all $X, Y, Z_1, Z_2 \in T_p M^n$. If $\nabla h = 0$, we say that M^n has parallel second fundamental form or, for short, that it is a *parallel* hypersurface. If $R \cdot h = 0$, we say that M^n is a *semi-parallel* hypersurface.

For any integer $k \geq 2$, we define recursively

$$\begin{aligned} (\nabla^k h)(X_1, \dots, X_k, Y, Z) &= X_1[(\nabla^{k-1} h)(X_2, \dots, X_k, Y, Z)] \\ &\quad - (\nabla^{k-1} h)(\nabla_{X_1} X_2, \dots, X_k, Y, Z) - \dots - (\nabla^{k-1} h)(X_2, \dots, X_k, Y, \nabla_{X_1} Z) \end{aligned}$$

for $X_1, \dots, X_k, Y, Z \in T_p M^n$. We call a hypersurface satisfying $\nabla^k h = 0$ a *k-parallel* hypersurface or a *higher order parallel* hypersurface. With slight modifications, all these notions can also be defined for submanifolds with arbitrary codimension.

The classification of parallel hypersurfaces in real space forms is proven in [13], whereas for the classification of *k-parallel* hypersurfaces in real space forms we refer to [7], [8] and [9]:

Theorem 2. [13] *A parallel hypersurface in a simply connected, complete real space form of constant sectional curvature c is one of the following. In \mathbb{E}^{n+1} : an open part of a product immersion $\mathbb{E}^k \times \mathbb{S}^{n-k}$, $k \in \{0, \dots, n\}$. In $\mathbb{S}^{n+1}(c)$: an open part of a product immersion $\mathbb{S}^k \times \mathbb{S}^{n-k}$, $k \in \{0, \dots, n\}$. In $\mathbb{H}^{n+1}(c)$: an open part of a product immersion $\mathbb{H}^k \times \mathbb{S}^{n-k}$, $k \in \{0, \dots, n\}$ or of a horosphere.*

Theorem 3. [7], [8], [9] *A k -parallel hypersurface in a simply connected, complete real space form of constant sectional curvature c is one of the following. In \mathbb{E}^{n+1} : an open part of a parallel hypersurface or of a cylinder on a plane curve, whose curvature is a polynomial function of degree at most $k - 1$ of the arc length. In $\mathbb{S}^{n+1}(c)$: an open part of a parallel hypersurface or, for $n = 2$, of the inverse image under the Hopf-fibration $\mathbb{S}^3(c) \rightarrow \mathbb{S}^2(4c)$ of a spherical curve in $\mathbb{S}^2(4c)$ whose geodesic curvature is a polynomial of degree at most $k - 1$ of the arc length. In $\mathbb{H}^{n+1}(c)$: an open part of a parallel hypersurface.*

In [2] the following classification for parallel surfaces in BCV spaces is proven:

Theorem 4. [2] *A parallel surface in $\widetilde{M}^3(\kappa, \tau)$, with $\kappa \neq 4\tau^2$, is an open part of a Hopf cylinder over a Riemannian circle in $\widetilde{M}^2(\kappa)$ or of a totally geodesic leaf of the Hopf fibration, the latter case only occuring for $\tau = 0$.*

The technique used in the proof of this theorem is based on the fact that for parallel surfaces the left-hand side of Codazzi's equation (18) is zero. For k -parallel surfaces another approach is needed.

We refer to [10] for a proof of the following lemma:

Lemma 1. [10] *A k -parallel surface immersed in a three-dimensional Riemannian manifold is semi-parallel, or equivalently, it is flat or totally umbilical.*

This means that in our search for k -parallel surfaces in BCV spaces, we can focus on totally umbilical surfaces (meaning that at every point the shape operator is a scalar multiple of the identity) and flat surfaces (meaning that the Gaussian curvature at every point is zero). In the next section we will give a complete classification of totally umbilical surfaces in BCV spaces and in the last section we will classify all flat, k -parallel surfaces in BCV spaces.

5. TOTALLY UMBILICAL SURFACES

In [16], it was proven that there are no totally umbilical surfaces in the Heisenberg group Nil_3 . The following lemma generalizes this result.

Lemma 2. *Let $M^2 \rightarrow \widetilde{M}^3(\kappa, \tau)$ be a totally umbilical surface with shape operator $S = \lambda \text{id}$. Then $\tau = 0$ and the following equations hold:*

$$(22) \quad T[\lambda] = -\kappa \cos \theta \sin^2 \theta, \quad (JT)[\lambda] = 0, \quad T[\theta] = \lambda \sin \theta, \quad (JT)[\theta] = 0,$$

$$(23) \quad \nabla_T T = \lambda \cos \theta T, \quad \nabla_{JT} T = \lambda \cos \theta JT, \quad \nabla_T JT = \lambda \cos \theta JT, \quad \nabla_{JT} JT = -\lambda \cos \theta T.$$

Proof. First assume that θ is identically zero. Then with the notations of section 2 we have $TM^2 = \text{span}\{e_1, e_2\}$. But according to Frobenius' theorem and (10), this distribution is only integrable if $\tau = 0$. Now $T = JT = 0$ and, since $Se_1 = -\widetilde{\nabla}_{e_1} e_3 = 0$ and $Se_2 = -\widetilde{\nabla}_{e_2} e_3 = 0$, also $\lambda = 0$. All equations stated in the lemma are satisfied.

We now work on an open subset of M^2 where θ is nowhere zero. From Codazzi's equation (18) for $X = T$ and $Y = JT$, we get

$$(24) \quad T[\lambda] = -(\kappa - 4\tau^2) \cos \theta \sin^2 \theta, \quad JT[\lambda] = 0.$$

The structure equations (20) and (21) yield

$$(25) \quad \nabla_T T = \cos \theta (\lambda T - \tau JT), \quad \nabla_{JT} T = \cos \theta (\tau T + \lambda JT), \quad T[\theta] = \lambda \sin \theta, \quad (JT)[\theta] = \tau \sin \theta.$$

Using orthonormal expansion and $\langle T, JT \rangle = 0$, $\langle T, T \rangle = \langle JT, JT \rangle = \sin^2 \theta$, we get

$$(26) \quad \nabla_T JT = \cos \theta (\tau T + \lambda JT), \quad \nabla_{JT} JT = \cos \theta (-\lambda T + \tau JT).$$

Remark that $[T, JT] = \nabla_T JT - \nabla_{JT} T = 0$ and hence

$$0 = [T, JT][\lambda] = T[(JT)[\lambda]] - (JT)[T[\lambda]] = (\kappa - 4\tau^2) \tau \sin^2 \theta (2 \cos^2 \theta - \sin^2 \theta).$$

Since we assume $\kappa - 4\tau^2 \neq 0$ and $\sin \theta \neq 0$, either $\tau = 0$ or $2 \cos^2 \theta - \sin^2 \theta = 0$. But the latter implies that θ is a constant and then from the last equation of (25) we also get $\tau = 0$. The

equations stated in the lemma follow easily from (24), (25) and (26). By a continuity argument, these will hold on the whole of M^2 . \square

The following is an immediate corollary of Lemma 2.

Theorem 5. *The only BCV spaces admitting totally umbilical surfaces are the Riemannian products $(\mathbb{S}^2(\kappa) \setminus \{\infty\}) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$.*

It is now sufficient to study totally umbilical surfaces in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$. To do this, we consider these spaces as hypersurfaces of the four-dimensional Euclidean space \mathbb{E}^4 and the four-dimensional Lorentzian space $\mathbb{L}^4 = (\mathbb{R}^4, -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)$ respectively:

$$\mathbb{S}^2(\kappa) \times \mathbb{R} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{E}^4 \mid x_1^2 + x_2^2 + x_3^2 = \frac{1}{\kappa} \right\}$$

and

$$\mathbb{H}^2(\kappa) \times \mathbb{R} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{L}^4 \mid -x_1^2 + x_2^2 + x_3^2 = \frac{1}{\kappa}, x_1 > 0 \right\}.$$

Remark that in both cases the vector field $\tilde{\xi}$, defined by $\tilde{\xi}(x_1, x_2, x_3, x_4) = \sqrt{|\kappa|}(x_1, x_2, x_3, 0)$, is orthogonal to the hypersurface and that $\langle \tilde{\xi}, \tilde{\xi} \rangle = 1$ in the first case and $\langle \tilde{\xi}, \tilde{\xi} \rangle = -1$ in the second case.

First, we remark that the only totally umbilical surfaces that are also higher order parallel are trivial:

Proposition 1. *A k -parallel, totally umbilical surface in $\mathbb{S}^2(\kappa) \times \mathbb{R}$, respectively $\mathbb{H}^2(\kappa) \times \mathbb{R}$, is totally geodesic and an open part of $\mathbb{S}^2(\kappa) \times \{t_0\}$ or $\mathbb{S}^1(\kappa) \times \mathbb{R}$, respectively $\mathbb{H}^2(\kappa) \times \{t_0\}$ or $\mathbb{H}^1(\kappa) \times \mathbb{R}$. Moreover, these surfaces are the only totally geodesic ones.*

Proof. If θ is identically zero, the surface is an open part of $\mathbb{S}^2(\kappa) \times \{t_0\}$ or $\mathbb{H}^2(\kappa) \times \{t_0\}$. Hence we may assume that $\theta \neq 0$. Putting $U = \frac{T}{\|T\|} = \frac{T}{\sin \theta}$ and $V = JT$, we have $[U, V] = 0$, so we can take coordinates (u, v) with $U = \frac{\partial}{\partial u}$ and $V = \frac{\partial}{\partial v}$. Remark that λ and θ only depend on u and

$$(27) \quad \lambda' = -\kappa \cos \theta \sin \theta = -\frac{\kappa}{2} \sin(2\theta), \quad \theta' = \lambda.$$

Since

$$\nabla_U U = \frac{1}{\sin \theta} \left(T \left[\frac{1}{\sin \theta} \right] T + \frac{1}{\sin \theta} \nabla_T T \right) = 0,$$

we have

$$0 = (\nabla^k h)(U, U, \dots, U, U) = U[U[\dots U[h(U, U)] \dots]] = \lambda^{(k)}(u),$$

which implies that λ is a polynomial of degree at most $k-1$ in u . Now from (27), we see that both $\sin(2\theta)$ and θ are polynomials in u . The only possibility is that θ is a constant and thus, again from (27), $\lambda = 0$ and $\cos \theta = 0$. So $\theta = \frac{\pi}{2}$ and the surface is an open part of $\gamma \times \mathbb{R}$, with γ a curve in $\mathbb{S}^2(\kappa)$ or $\mathbb{H}^2(\kappa)$.

It remains to prove that γ is a geodesic. We continue the proof for $\kappa > 0$, but the other case is completely similar. Assume that γ is parametrized by arc length and denote the immersion by

$$F : M^2 \rightarrow \mathbb{S}^2(\kappa) \times \mathbb{R} \subset \mathbb{E}^4 : (s, t) \mapsto (\gamma(s), t).$$

Denoting by “ \cdot ” the inner product on \mathbb{E}^3 and by “ \times ” the cross product, we have that $F_s = (\gamma', 0)$ and $F_t = (0, 1)$ span the tangent space, that $\tilde{\xi} = \sqrt{\kappa}(\gamma, 0)$ is a unit vector orthogonal to the surface and orthogonal to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\xi = \sqrt{\kappa}(\gamma \times \gamma', 0)$ is a unit vector orthogonal to M^2 , tangent to $\mathbb{S}^2(\kappa) \times \mathbb{R}$. Moreover

$$\begin{aligned} \left\langle S \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right\rangle &= \langle F_{ss}, \xi \rangle = \kappa((\gamma \times \gamma') \cdot \gamma'', 0), \\ \left\langle S \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right\rangle &= \langle F_{st}, \xi \rangle = (0, 0), \\ \left\langle S \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle &= \langle F_{tt}, \xi \rangle = (0, 0), \end{aligned}$$

and thus the surface is totally umbilical (and automatically totally geodesic) if and only if $(\gamma \times \gamma') \cdot \gamma'' = 0$, or equivalently, if and only if γ'' is proportional to γ . This means that γ'' has no component tangent to $\mathbb{S}^2(\kappa)$ and hence has to be a geodesic, i.e. a great circle.

The fact that these surfaces are the only totally geodesic ones follows immediately from the first equation of (22). \square

Before proceeding with the full classification, we develop some machinery to study surfaces in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$.

Consider an isometric immersion $F : M^2 \rightarrow \mathbb{S}^2(\kappa) \times \mathbb{R}$. Denoting by ξ a unit vector tangent to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and normal to M^2 , one easily sees that the fourth components of F_*T , F_*JT and ξ in \mathbb{E}^4 satisfy

$$(28) \quad (F_*T)_4 = \sin^2 \theta, \quad (F_*JT)_4 = 0, \quad \xi_4 = \cos \theta.$$

Take $\tilde{\xi}$ as above and let X be a tangent vector to M^2 . Then $\langle \nabla_X^\perp \tilde{\xi}, \xi \rangle = \langle D_X \tilde{\xi}, \xi \rangle = X_1 \xi_1 + X_2 \xi_2 + X_3 \xi_3 = -X_4 \xi_4 = -\langle X, T \rangle \cos \theta$, where D denotes the Euclidean connection. Thus, the normal connection of M^2 as a submanifold of \mathbb{E}^4 is given by

$$\nabla_X^\perp \tilde{\xi} = -\langle X, T \rangle \cos \theta \xi, \quad \nabla_X^\perp \xi = \langle X, T \rangle \cos \theta \tilde{\xi}.$$

Using Weingarten's formula, we see that the shape operator associated to $\tilde{\xi}$, which we denote by \tilde{S} , must satisfy

$$\begin{aligned} F_*(\tilde{S}T) &= (-(F_*T)_1, -(F_*T)_2, -(F_*T)_3, 0) - \cos \theta \sin^2 \theta (\xi_1, \xi_2, \xi_3, \cos \theta), \\ F_*(\tilde{S}(JT)) &= (-(F_*JT)_1, -(F_*JT)_2, -(F_*JT)_3, 0) = -JT. \end{aligned}$$

The second equation implies that the matrix of \tilde{S} with respect to the basis $\{T, JT\}$ takes the form

$$\tilde{S} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

and looking at the fourth component of the first equation we get $a = -\cos^2 \theta$. Hence

$$(29) \quad \tilde{S} = \begin{pmatrix} -\cos^2 \theta & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark that from the other components of the first equation

$$(30) \quad (F_*T)_j = -\cos \theta \xi_j, \quad j = 1, 2, 3.$$

We can do exactly the same for $\mathbb{H}^2(\kappa) \times \mathbb{R}$. The equations (28) remain the same. The normal connection changes to

$$\nabla_X^\perp \tilde{\xi} = -\langle X, T \rangle \cos \theta \xi, \quad \nabla_X^\perp \xi = -\langle X, T \rangle \cos \theta \tilde{\xi},$$

but the shape operator associated to $\tilde{\xi}$, (29), and formula (30) remain the same.

We will now classify totally umbilical surfaces in $\mathbb{S}^2(1) \times \mathbb{R}$ and $\mathbb{H}^2(-1) \times \mathbb{R}$ and for arbitrary κ the totally umbilical surfaces will then be homothetic to these.

Theorem 6. *Let $F : M^2 \rightarrow \mathbb{S}^2(1) \times \mathbb{R} \subset \mathbb{E}^4$ be a totally umbilical surface with shape operator $S = \lambda \text{id}$ and angle function θ , which is not totally geodesic. Then one can choose local coordinates (u, v) on M^2 such that λ and θ only depend on u and*

$$(31) \quad \theta(u) = \arctan \left(\frac{2ce^{\pm cu}}{1 - c^2 + e^{\pm 2cu}} \right), \quad \lambda(u) = \frac{\theta'(u)}{\sin \theta(u)},$$

for some real constant $c > 0$. Moreover, the immersion is, up to an isometry, locally given by

$$(32) \quad F(u, v) = \frac{1}{c} \left(\lambda, \sin \theta \cos v, \sin \theta \sin v, c \int \sin^2 \theta du \right).$$

Proof. It follows from (23) that $[T, JT] = 0$. Hence, we can take local coordinates (u, v) on M^2 , such that $T = \frac{\partial}{\partial u}$, $JT = \frac{\partial}{\partial v}$. From (22) we see that λ and θ only depend on u and that they satisfy

$$(33) \quad \lambda^2 + \sin^2 \theta = c^2, \quad \theta' = \lambda \sin \theta,$$

for some strictly positive real constant c .

From the formula of Gauss, (22), (29) and (30), we obtain for $j = 1, 2, 3$

$$(34) \quad (F_j)_{uu} = \lambda \cos \theta (F_j)_u - \lambda \frac{\sin^2 \theta}{\cos \theta} (F_j)_u - \cos^2 \theta \sin^2 \theta F_j,$$

$$(35) \quad (F_j)_{uv} = \lambda \cos \theta (F_j)_v,$$

$$(36) \quad (F_j)_{vv} = -\frac{\lambda}{\cos \theta} (F_j)_u - \sin^2 \theta F_j.$$

The equations for the fourth component are trivially satisfied. The solution of (35) is

$$(37) \quad F_j = (A_j(u) + B_j(v)) \exp \left(\int \lambda \cos \theta du \right),$$

where A_j and B_j are real-valued functions in one variable. Substituting this in (34) yields

$$(38) \quad A_j = a_j \int \exp \left(- \int \frac{\lambda}{\cos \theta} du \right) du + \alpha_j$$

with $a_j, \alpha_j \in \mathbb{R}$, and substituting it in (36) gives $B_j'' + (\lambda^2 + \sin^2 \theta) B_j = A_j'' - (\lambda^2 + \sin^2 \theta) A_j$, or equivalently $B_j'' + c^2 B_j = A_j'' - c^2 A_j$. It is easy to check that the right hand side of this equation is constant and thus the solution for B_j is

$$(39) \quad B_j = b_j \cos(cv) + \beta_j \sin(cv) + \frac{A_j''}{c^2} - A_j,$$

with $b_j, \beta_j \in \mathbb{R}$. By substituting (38) and (39) in (37), we conclude that the functions F_j take the form

$$(40) \quad F_j = \left(-a_j \frac{\lambda}{c^2 \cos \theta} \exp \left(- \int \frac{\lambda}{\cos \theta} du \right) + b_j \cos(cv) \right. \\ \left. + \beta_j \sin(cv) \right) \exp \left(\int \lambda \cos \theta du \right), \quad j = 1, 2, 3$$

and from (28):

$$(41) \quad F_4 = \int \sin^2 \theta du.$$

There are some conditions on F which we have neglected so far, namely $F \in \mathbb{S}^2(1) \times \mathbb{R}$, $\langle \xi, F_u \rangle = \langle \xi, F_v \rangle = 0$, $\langle \tilde{\xi}, F_u \rangle = \langle \tilde{\xi}, F_v \rangle = 0$, $\langle F_u, F_u \rangle = \langle F_v, F_v \rangle = \sin^2 \theta$, $\langle \xi, \xi \rangle = \langle \tilde{\xi}, \tilde{\xi} \rangle = 1$ and $\langle \xi, \tilde{\xi} \rangle = \langle F_u, F_v \rangle = 0$. These are equivalent to

$$(42) \quad \sum_{j=1}^3 F_j^2 = 1, \quad \sum_{j=1}^3 (F_j)_u^2 = \cos^2 \theta \sin^2 \theta, \quad \sum_{j=1}^3 (F_j)_v^2 = \sin^2 \theta, \quad \sum_{j=1}^3 (F_j)_u (F_j)_v = 0.$$

Now looking at $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ as vectors in \mathbb{R}^3 with the Euclidean inner product “ \cdot ”, the conditions (42) are equivalent to

$$a \cdot b = a \cdot \beta = b \cdot \beta = 0,$$

$$\|a\|^2 = a \cdot a = c^2 \cos^2 \theta \exp \left(2 \int \frac{\lambda \sin^2 \theta}{\cos \theta} du \right),$$

$$\|b\|^2 = b \cdot b = \beta \cdot \beta = \frac{\sin^2 \theta}{c^2} \exp \left(-2 \int \lambda \cos \theta du \right).$$

Remark that the right hand sides of these equations are constant. They imply that after a suitable isometry of $\mathbb{S}^2(1) \times \mathbb{R}$ we may assume that

$$\begin{aligned} a &= \left(-c \cos \theta \exp \left(\int \frac{\lambda \sin^2 \theta}{\cos \theta} du \right), 0, 0 \right), \\ b &= \left(0, \frac{\sin \theta}{c} \exp \left(- \int \lambda \cos \theta du \right), 0 \right), \\ \beta &= \left(0, 0, \frac{\sin \theta}{c} \exp \left(- \int \lambda \cos \theta du \right) \right). \end{aligned}$$

Now the reparametrization $cv \mapsto v$ gives the result (32).

To conclude, we solve the equations (33) explicitly. Putting $\theta = \arctan(f)$, we obtain

$$\left(\frac{\theta'}{\sin \theta} \right)^2 + \sin^2 \theta = c^2 \Leftrightarrow \left(\frac{f'}{f\sqrt{1+f^2}} \right)^2 + \frac{f^2}{1+f^2} = c^2 \Leftrightarrow \frac{(f')^2}{f^2(c^2 + (c^2 - 1)f^2)} = 1.$$

From the last equation we see that $c^2 + (c^2 - 1)f^2$ has to be positive and hence we can proceed by integration:

$$\begin{aligned} \frac{f'}{f\sqrt{c^2 + (c^2 - 1)f^2}} = \pm 1 &\Leftrightarrow \ln \left(\frac{c + \sqrt{c^2 + (c^2 - 1)f^2}}{f} \right) = \pm cu + d \\ &\Leftrightarrow f = \frac{2ce^{\pm cu + d}}{1 - c^2 + e^{2(\pm cu + d)}}, \end{aligned}$$

for some $d \in \mathbb{R}$. After a change of the u -coordinate, which does not change $\frac{\partial}{\partial u}$, we obtain the result (31). \square

Remark 1. We can write (32) in a more explicit form. After the reparametrization $e^{\pm cu} \mapsto u$ and, if necessary, an isometry switching the sign of some of the components, (32) is given by

$$F(u, v) = \left(\frac{2u \cos v}{p(u)q(u)}, \frac{2u \sin v}{p(u)q(u)}, \frac{1 - c^2 - u^2}{p(u)q(u)}, \ln \left(\frac{p(u)}{q(u)} \right) \right),$$

where $p(u) = \sqrt{u^2 + (c - 1)^2}$ and $q(u) = \sqrt{u^2 + (c + 1)^2}$.

Theorem 7. *Let $F : M^2 \rightarrow \mathbb{H}^2(-1) \times \mathbb{R} \subset \mathbb{E}_1^4$ be a totally umbilical surface with shape operator $S = \lambda \text{id}$ and angle function θ , which is not totally geodesic. Then one can choose local coordinates (u, v) on M^2 such that λ and θ only depend on u and we are in one of the following three cases:*

(i) $\theta(u)$ and $\lambda(u)$ are given by

$$(43) \quad \theta(u) = \arctan \left(\frac{2ce^{\pm cu}}{1 + c^2 - e^{\pm 2cu}} \right), \quad \lambda(u) = \frac{\theta'(u)}{\sin \theta(u)},$$

for some real constant $c > 0$, and the immersion is, up to an isometry, locally given by

$$(44) \quad F(u, v) = \frac{1}{c} \left(\lambda, \sin \theta \cos v, \sin \theta \sin v, c \int \sin^2 \theta du \right),$$

(ii) $\theta(u)$ and $\lambda(u)$ are given by

$$(45) \quad \theta(u) = \text{arccot}(\pm u), \quad \lambda(u) = \frac{\mp 1}{\sqrt{1 + u^2}},$$

and the immersion is, up to an isometry, locally given by

$$(46) \quad F(u, v) = \frac{1}{\sqrt{1 + u^2}} \left(\frac{u^2 + v^2}{2} + 1, v, \frac{u^2 + v^2}{2}, \sqrt{1 + u^2} \arctan u \right),$$

(iii) $\theta(u)$ and $\lambda(u)$ are given by

$$(47) \quad \theta(u) = \arctan\left(\frac{\tan c}{\sin(\pm u \sin c)}\right), \quad \lambda(u) = \frac{\theta'(u)}{\sin \theta(u)},$$

for some real constant $c \neq 0$, and the immersion is, up to an isometry, locally given by

$$(48) \quad F(u, v) = \frac{1}{\sin c} \left(\sin \theta \cosh v, \sin \theta \sinh v, \lambda, \sin c \int \sin^2 \theta du \right).$$

Proof. We use again the coordinates (u, v) such that $T = \frac{\partial}{\partial u}u$ and $JT = \frac{\partial}{\partial v}$. From (22), we obtain that λ and θ only depend on u and that they satisfy $\lambda^2 - \sin^2 \theta = C$, $\theta' = \lambda \sin \theta$, for some real constant $C > -1$. The formula of Gauss yields for $j = 1, 2, 3$:

$$\begin{aligned} (F_j)_{uu} &= \lambda \cos \theta (F_j)_u - \lambda \frac{\sin^2 \theta}{\cos \theta} (F_j)_u + \cos^2 \theta \sin^2 \theta F_j, \\ (F_j)_{uv} &= \lambda \cos \theta (F_j)_v, \\ (F_j)_{vv} &= -\frac{\lambda}{\cos \theta} (F_j)_u + \sin^2 \theta F_j, \end{aligned}$$

such that F_j again takes the form (37), with A_j again equal to (38). The differential equation for B_j becomes $B_j'' + (\lambda^2 - \sin^2 \theta)B_j = A_j'' - (\lambda^2 - \sin^2 \theta)A_j$, or equivalently $B_j'' + CB_j = A_j'' - CA_j$. The right hand side of this equation is again constant. We now consider three cases.

Case (A): $C > 0$. This case corresponds to the first case of the theorem. We can put $C = c^2$ for some strictly positive real constant c . The rest of the proof is similar to the one above and we will therefore omit it.

Case (B): $C = 0$. The solution of the equations $\lambda^2 = \sin^2 \theta$ and $\theta' = \lambda \sin \theta$ is given by (45). Substituting this in (38) yields that A_j takes the form $A_j(u) = p_j u^2 + q_j$ for some $p_j, q_j \in \mathbb{R}$. The equation for B_j becomes $B_j'' = A_j''$. From this equation and (37), we obtain

$$F_j = \frac{1}{\sqrt{1+u^2}} (a_j(u^2 + v^2) + b_j v + c_j), \quad j = 1, 2, 3,$$

where $a_j, b_j, c_j \in \mathbb{R}$. Moreover, from (28) and (45), we have

$$F_4 = \int \sin^2 \theta du = \arctan u.$$

The conditions analogous to (42) now read

$$(49) \quad \begin{aligned} -F_1^2 + F_2^2 + F_3^2 &= -1, \\ -(F_1)_u^2 + (F_2)_u^2 + (F_3)_u^2 &= \cos^2 \theta \sin^2 \theta, \\ -(F_1)_v^2 + (F_2)_v^2 + (F_3)_v^2 &= \sin^2 \theta, \\ -(F_1)_u(F_1)_v + (F_2)_u(F_2)_v + (F_3)_u(F_3)_v &= 0, \end{aligned}$$

and looking at a, b and c as vectors in \mathbb{R}^3 , but now equipped with the standard Lorentzian inner product “ \cdot ”, these are equivalent to $a \cdot a = a \cdot b = b \cdot c = 0$, $a \cdot c = -\frac{1}{2}$, $b \cdot b = 1$, $c \cdot c = -1$. After a suitable isometry of $\mathbb{H}^2(-1) \times \mathbb{R}$, we may assume that $a = (\frac{1}{2}, 0, \frac{1}{2})$, $b = (0, 1, 0)$ and $c = (1, 0, 0)$. This gives the result (46).

Case (C): $C < 0$. Clearly, we have $C > -1$ and hence we may put $C = -\sin^2 c$, for some real number c . The equation for B_j becomes $B_j'' - \sin^2 c B_j = A_j'' + \sin^2 c A_j$, with solution

$$(50) \quad B_j = b_j \cosh(v \sin c) + \beta_j \sinh(v \sin c) - \frac{A_j''}{\sin^2 c} - A_j.$$

Hence F is given by

$$(51) \quad F_j = \left(-a_j \frac{\lambda}{(\sin^2 c) \cos \theta} \exp \left(- \int \frac{\lambda}{\cos \theta} du \right) + b_j \cosh(v \sin c) \right. \\ \left. + \beta_j \sinh(v \sin c) \right) \exp \left(\int \lambda \cos \theta du \right), \quad j = 1, 2, 3$$

and F_4 takes the form (41).

Looking at a , b and β as vectors in \mathbb{R}^3 with the standard Lorentzian inner product, the conditions (49) yield

$$\begin{aligned} a \cdot b &= a \cdot \beta = b \cdot \beta = 0, \\ a \cdot a &= \sin^2 c \cos^2 \theta \exp \left(2 \int \frac{\lambda \sin^2 \theta}{\cos \theta} du \right), \\ b \cdot b &= -\beta \cdot \beta = -\frac{\sin^2 \theta}{\sin^2 c} \exp \left(-2 \int \lambda \cos \theta du \right), \end{aligned}$$

Remark that the right hand sides are again constant and that b is a timelike vector, whereas a and β are spacelike. A suitable isometry of $\mathbb{H}^2(-1) \times \mathbb{R}$, followed by the reparametrization $v \sin c \mapsto v$, transforms the immersion given by (51) and (41) into (48).

Finally, we solve the equations $\lambda^2 - \sin^2 \theta = -\sin^2 c$, $\theta' = \lambda \sin \theta$ explicitly. Putting $\theta = \arctan(f)$, we obtain

$$\left(\frac{\theta'}{\sin \theta} \right)^2 - \sin^2 \theta = -\sin^2 c \Leftrightarrow \frac{(f')^2}{f^2(f^2 \cos^2 c - \sin^2 c)} = 1.$$

We see that $f^2 \cos^2 c - \sin^2 c > 0$, and by integration, we obtain

$$\arctan \left(\frac{\sin c}{\sqrt{f^2 \cos^2 c - \sin^2 c}} \right) = \pm u \sin c + d \Leftrightarrow f = \frac{\tan c}{\sin(\pm u \sin c + d)},$$

for some $d \in \mathbb{R}$. After a translation in the u -coordinate, we obtain (48). \square

Remark 2. We can write the immersions of the first and the last case of Theorem 7 more explicitly. After the substitution $e^{\pm cu} \mapsto u$, the immersion (44) becomes

$$F(u, v) = \left(\frac{1 + c^2 + u^2}{p(u)q(u)}, \frac{2u \cos v}{p(u)q(u)}, \frac{2u \sin v}{p(u)q(u)}, \frac{1}{4c^2} \arctan \left(\frac{u^2 - 1 + c^2}{2c} \right) \right),$$

with $p(u) = \sqrt{(u-1)^2 + c^2}$ and $q(u) = \sqrt{(u+1)^2 + c^2}$.

The immersion (48) is, after the substitution $\pm u \sin c \mapsto u$, given by

$$F(u, v) = \left(\frac{\cosh v}{p(u)}, \frac{\sinh v}{p(u)}, \frac{-\cos c \cos u}{p(u)}, \arctan \left(\frac{\tan u}{\sin c} \right) \right),$$

with $p(u) = \sqrt{1 - \cos^2 c \cos^2 u}$.

Remark 3. Totally umbilical surfaces in BCV spaces and in the Lie group Sol_3 were independently studied in [17], from a global viewpoint.

6. HIGHER ORDER PARALLEL SURFACES

The following example shows that every Hopf-cylinder in a BCV space is flat.

Example 1. Consider a Hopf-cylinder in $\widetilde{M}^3(\kappa, \tau)$. Let $\{E_1 = ae_1 + be_2, E_2 = e_3\}$, with $a^2 + b^2 = 1$, be an orthonormal frame field along the surface, then $N = E_1 \times E_2 = be_1 - ae_2$ is a unit normal. Using the equations in (11), one computes

$$\begin{aligned} \widetilde{\nabla}_{E_1} N &= \left(aE_1[b] - bE_1[a] + \frac{\kappa}{2}(ay - bx) \right) E_1 - \tau E_2, \\ \widetilde{\nabla}_{E_2} N &= (aE_2[b] - bE_2[a] - \tau)E_1. \end{aligned}$$

This means that the shape operator with respect to the basis $\{E_1, E_2\}$ takes the form

$$\begin{aligned} S &= \begin{pmatrix} -aE_1[b] + bE_1[a] - \frac{\kappa}{2}(ay - bx) & -aE_2[b] + bE_2[a] + \tau \\ \tau & 0 \end{pmatrix} \\ &= \begin{pmatrix} -aE_1[b] + bE_1[a] - \frac{\kappa}{2}(ay - bx) & \tau \\ \tau & 0 \end{pmatrix}, \end{aligned}$$

the last equation due to the symmetry. Remark that from this symmetry we have $aE_2[b] = bE_2[a]$, which, together with $a^2 + b^2 = 1$ implies that a and b are constant along the fibres of the Hopf-fibration. From Gauss' equation (19), we have

$$K = \det S + \tau^2 + (\kappa - 4\tau^2) \cos^2 \theta = -\tau^2 + \tau^2 + (\kappa - 4\tau^2) \cos^2 \frac{\pi}{2} = 0.$$

Now consider an arbitrary flat surface M^2 in $\widetilde{M}^3(\kappa, \tau)$. Every $p \in M^2$ has an open neighbourhood U , which is isometric to an open part of \mathbb{E}^2 . Denote by (u, v) the Euclidean coordinates on U . Suppose $T = T_1 \frac{\partial}{\partial u} + T_2 \frac{\partial}{\partial v}$ and $S = (S_{ij})_{1 \leq i, j \leq 2}$ with respect to the orthonormal basis $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$. We consider $S_{11}, S_{12}, S_{22}, \cos \theta, T_1$ and T_2 as functions of the Euclidean coordinates (u, v) on U .

Lemma 3. *The functions $S_{11}, S_{12}, S_{22}, \cos \theta, T_1$ and T_2 satisfy the following system of equations:*

$$(52) \quad T_1^2 + T_2^2 + \cos^2 \theta = 1;$$

$$(53) \quad S_{11}S_{22} - S_{12}^2 + \tau^2 + (\kappa - 4\tau^2) \cos^2 \theta = 0;$$

$$(54) \quad \frac{\partial S_{12}}{\partial u} - \frac{\partial S_{11}}{\partial v} = (\kappa - 4\tau^2)T_2 \cos \theta,$$

$$\frac{\partial S_{22}}{\partial u} - \frac{\partial S_{12}}{\partial v} = -(\kappa - 4\tau^2)T_1 \cos \theta;$$

$$(55) \quad \frac{\partial T_1}{\partial u} = S_{11} \cos \theta, \quad \frac{\partial T_1}{\partial v} = (S_{12} + \tau) \cos \theta,$$

$$\frac{\partial T_2}{\partial u} = (S_{12} - \tau) \cos \theta, \quad \frac{\partial T_2}{\partial v} = S_{22} \cos \theta;$$

$$(56) \quad \frac{\partial \cos \theta}{\partial u} = -S_{11}T_1 - S_{12}T_2 + \tau T_2,$$

$$\frac{\partial \cos \theta}{\partial v} = -S_{12}T_1 - S_{22}T_2 - \tau T_1.$$

Proof. Equation (52) follows immediately from the definitions of T and θ . Equation (53) expresses Gauss' equation (19), while the equations (54) express the equation of Codazzi (18). The equations in (55) and (56) are nothing but the structure equations (20) and (21). \square

The following result is the last step to obtain a full classification of higher order parallel surfaces in BCV spaces.

Proposition 2. *A k -parallel, flat surface M^2 in a BCV space $\widetilde{M}^3(\kappa, \tau)$, with $\kappa \neq 4\tau^2$, is an open part of a Hopf-cylinder over a curve in $\widetilde{M}^2(\kappa)$, whose curvature is a polynomial function of degree at most $k - 1$ of the arc length.*

Proof. Since M^2 is k -parallel and flat, the functions S_{11}, S_{12} and S_{22} have to be polynomials of degree at most $k - 1$ in u and v . First one can show that the equations in lemma 3 then imply that θ has to be a constant. This proof is very similar to the proof of the Main Theorem in [10] and we will therefore omit it.

Now it follows from (54) that the functions T_1 and T_2 are polynomial functions in u and v . Since T_1 and T_2 satisfy $T_1^2 + T_2^2 = 1 - \cos^2 \theta$ and θ is a constant, they have to be constant. Then the equations in (55) imply that either $\cos \theta = 0$ or $\tau = 0$ and $S = 0$. Totally geodesic surfaces in BCV-spaces with $\tau = 0$ are classified in proposition 1 and it is clear that the only flat ones are Hopf-cylinders. Hence we may conclude that M^2 is an open part of a Hopf-cylinder.

To finish, we prove the assertion about the curvature of the base curve. Taking E_1 and E_2 as in example 1, one can verify that $\nabla_{E_i} E_j = 0$ and hence we can take Euclidean coordinates (u, v) such that $E_1 = \frac{\partial}{\partial u}$ and $E_2 = \frac{\partial}{\partial v}$. As we remarked before, a and b will only depend on u and we

write a' and b' for the derivatives with respect to u . The base curve $\gamma(u) = (x(u), y(u))$ satisfies $\gamma' = \pi_* E_1 = (1 + \frac{\kappa}{4}(x^2 + y^2))(a, b)$, such that u is an arc length parameter. We compute

$$\kappa_\gamma = (1 + \frac{\kappa}{4}(x^2 + y^2)) \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{\frac{3}{2}}} + \frac{\kappa}{2} \frac{x'y - xy'}{((x')^2 + (y')^2)^{\frac{1}{2}}} = ab' - a'b + \frac{\kappa}{2}(ay - bx) = -S_{11}.$$

Looking at the expression for S , we see that the surface is k -parallel if and only if S_{11} is a polynomial of degree at most $k - 1$ in u and v . This is equivalent to κ_γ being a polynomial of degree at most $k - 1$ in u . \square

From Lemma 1, Proposition 1 and Proposition 2 we obtain a full classification of higher order parallel surfaces in 3-dimensional homogeneous spaces with 4-dimensional isometry group:

Theorem 8. *A k -parallel surface in a BCV space $\widetilde{M}^3(\kappa, \tau)$, with $\kappa \neq 4\tau^2$, is one of the following:*

- (i) *an open part of a Hopf-cylinder on a curve whose geodesic curvature is a polynomial function of degree at most $k - 1$ of the arc length;*
- (ii) *an open part of a totally geodesic leaf of the Hopf-fibration;*

the latter case only occuring when $\tau = 0$.

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